On the Rigidity Matroid of Highly Connected Graphs

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Abstract: We prove that for any positive integer d, the generic rigidity matroid $\mathcal{R}_d(G)$ uniquely determines the underlying graph G, provided that either G is $d(d+1)^2$-connected, or $\mathcal{R}_d(G)$ is (vertically) $(d(d+1))^2$-connected. This extends previous results for the $d = 1$ and $d = 2$ cases.

Keywords: rigidity matroid, graph isomorphism, vertex connectivity

1 Introduction

In this note, we investigate when the d-dimensional generic rigidity matroid $\mathcal{R}_d(G)$ of a graph G determines G. Whitney gave a complete answer to this question in the $d = 1$ case by showing that for any pair of graphs G and H, any isomorphism of the graphic matroids $\mathcal{R}_1(G)$ and $\mathcal{R}_1(H)$ arises from a 2-isomorphism of G and H. In particular, his result implies the following unique reconstructibility result for 3-connected graphs.

Theorem 1 *[10] Let G and H be graphs, and let $\psi : E(G) \to E(H)$ be an isomorphism of $\mathcal{R}_1(G)$ and $\mathcal{R}_1(H)$. If G is 3-connected and H is without isolated vertices, then ψ is induced by a graph isomorphism.*

Our goal is to prove an analogue of Theorem 1 for higher values of d. For convenience, let us say that a graph G is $\mathcal{R}_d(G)$*-reconstructible* if for every graph H without isolated vertices and every isomorphism $\psi : \mathcal{R}_d(G) \to \mathcal{R}_d(H)$, ψ is induced by a graph isomorphism $\varphi : G \to H$. (That is, $\psi(uv) = \varphi(u)\varphi(v)$ holds for every edge $e \in E(G)$.) Using this terminology, Theorem 1 says that every 3-connected graph is $\mathcal{R}_1$-reconstructible. The constant 3 is best possible: a cycle of length at least four is not $\mathcal{R}_1$-reconstructible, since its graphic matroid has automorphisms that are not induced by automorphisms of the cycle.

Jordán and Kaszanitzky proved the following reconstructbility result in the two-dimensional case.

Theorem 2 *[5] Every 7-connected graph is $\mathcal{R}_2$-reconstructible.*

The proof of Theorem 2 relies on the fact that every 6-connected graph is 2-rigid, a classical result of Lovász and Yemini [6]. In fact, it is not difficult to see that a statement of the form "every c-connected graph is $\mathcal{R}_d$-reconstructible" also implies the statement "every c-connected graph is d-rigid." Since for the theorem of Lovász and Yemini, the constant 6 is known to be best possible, it follows that the constant in Theorem 2 cannot be improved beyond 6. It is open whether the constant 7 is best possible.

In a recent breakthrough, Villányi proved the following d-dimensional analogue of the theorem of Lovász and Yemini.

Theorem 3 *[9, Theorem 1.1] Every $d(d+1)$-connected graph is d-rigid.*

In this note, we use Theorem 3 to derive a d-dimensional analogue of Theorems 1 and 2.

Theorem 4 *Every $d(d+1)^2$-connected graph is $\mathcal{R}_d$-reconstructible.*

The constant $d(d+1)^2$ is probably far from optimal. In particular, it remains open whether every $d(d+1)$-connected graph is $\mathcal{R}_d$-reconstructible.

Our proof method also leads to a reconstructibility result where instead of assuming that the graph G is highly connected, we assume that the matroid $\mathcal{R}_d(G)$ is highly (vertically) connected.

Theorem 5 *Let G be a graph. If $\mathcal{R}_d(G)$ is (vertically) $(d(d+1))^2$-connected, then G is $\mathcal{R}_d$-reconstructible.*

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Theorem 5 answers positively a question of Brigitte and Herman Servatius [1, Problem 17].

Compared to previous Whitney-type reconstructibility results (such as Theorems 1 and 2 and results found in [3]), the main novelty of our proof is that it is not based on a combinatorial characterization of the rank function of the matroid in question. Indeed, finding such a (good) characterization for the d-dimensional generic rigidity matroid is a major open problem in the $d \geq 3$ cases.

We note that our proofs work for all 1-extendable abstract rigidity matroids. In fact, the ideas in this note are extended in [2] to show that for a rather general class of matroids, a Lovász-Yemini-type rigidity result (such as Theorem 3) leads to a Whitney-type reconstructibility result (such as Theorem 4).

2 Proof of main results

We assume basic knowledge of matroid theory and combinatorial rigidity theory. For references, see [4, 7, 8]. For a vertex v of a graph G, we shall let $\partial_G(v)$ denote the set of edges incident to v, while $d_G(v) = |\partial_G(v)|$ denotes the degree of v in G. The key concept behind our proof of Theorem 4 is the notion of vertical connectivity of matroids, which we recall below.

Let $\mathcal{M} = (E, r)$ be a matroid with rank function r and let k be a positive integer. We say that a bipartition (E_1, E_2) of E is a *vertical k-separation* of $\mathcal{M}$ if $r(E_1), r(E_2) \geq k$ and

$$r(E_1) + r(E_2) \leq r(E) + k - 1$$

holds. In this case $r(E_i) < r(E)$ for $i \in \{1, 2\}$. We say that $\mathcal{M}$ is *vertically k-connected* if $k \leq r(M)$ and $\mathcal{M}$ does not have vertical k'-separations for any positive integer $k' < k$. For $k = 1$, the latter condition is vacuous, and hence every matroid with positive rank is vertically 1-connected.

The following lemma shows that for certifying that a matroid is not vertically k-connected, it suffices to consider "vertical separations" $E = E_1 \cup E_2$ in which E_1 and E_2 are not necessarily disjoint.

Lemma 6 *Let $\mathcal{M} = (E, r)$ be a matroid, and let $E_1, E_2 \subseteq E$ be subsets of E, not necessarily disjoint, with $E = E_1 \cup E_2$. Suppose that $r(E_1), r(E_2) \geq k$ and $r(E_1) + r(E_2) \leq r(E) + k - 1$. Then $\mathcal{M}$ has a vertical k'-separation for some $k' \leq k$.*

Proof: Let $E_2' = E_2 - E_1$. Then (E_1, E_2') is a bipartition of E, and it is straightforward to check that it is a vertical k'-separation for $k' = k - (r(E_2) - r(E_2'))$. □

It is well-known that a graph without isolated vertices is k-connected if and only if its graphic matroid is vertically k-connected. The main ingredients to our proof of Theorem 4, Propositions 8 and 9 below, are partial generalizations of this fact to the d-dimensional generic rigidity matroid. Proposition 8 states that if $\mathcal{R}_d(G)$ has sufficiently high vertical connectivity, then G must have high vertex-connectivity (provided that it has no isolated vertices). Conversely, Proposition 9 shows that if G is highly vertex-redundantly rigid, then $\mathcal{R}_d(G)$ must have high vertical connectivity. Finally, by Theorem 3, high vertex-connectivity implies high vertex-redundant rigidity. By combining this triangle of implications with ideas from previous Whitney-type reconstructibility results, we can deduce that sufficiently highly connected graphs are $\mathcal{R}_d$-reconstructible.

The next lemma follows from [5, Theorem 4.2]. Since the proof is simple, we include it for completeness.

Lemma 7 *Let $G = (V, E)$ be a graph, and let $k \geq 2$ be an integer. If $\mathcal{R}_d(G)$ is vertically k-connected, then $|V| \geq k + d$.*

Proof: If $d = 1$, then by definition we have $k \leq r_d(G) \leq |V| - 1$. Thus we may assume $d \geq 2$. Note that since $\mathcal{R}_d(G)$ is vertically 2-connected, it is bridgeless and hence every vertex of G has degree at least $d+1$. Let $v \in V$ be a vertex of smallest degree in G, and let E_1 be a set of $d_G(v) - d + 1$ edges incident to v. Then $(E_1, E - E_1)$ is a bipartition of E with $r_d(E_1) = d_G(v) - d + 1$ and

$$r_d(E_1) + r_d(E - E_1) \leq d_G(v) - d + 1 + r_d(E) - 1 = r_d(E) + d_G(v) - d.$$

Let $w \in V - v$ be an arbitrary vertex of G. Then we have, using the fact that v has smallest degree, $r_d(E - E_1) \geq d_G(w) - 1 \geq d_G(v) - (d - 1)$. Thus $(E_1, E - E_1)$ is a vertical $(d(G) - d + 1)$-separation of $\mathcal{R}_d(G)$. Since $d(G) - d + 1 \leq |V| - d$, this implies that $|V| \geq k + d$, as desired. □

Proposition 8 *Let $G = (V, E)$ be a graph without isolated vertices, and let $k \geq d$ be an integer. If $\mathcal{R}_d(G)$ is vertically $(dk - \binom{d+1}{2} + 2)$-connected, then G is $(k+1)$-connected.*

Proof: Note that by Lemma 7, G has at least $k+2$ vertices. Furthermore, since $\mathcal{R}_d(G)$ is vertically 2-connected, G is a connected (in fact, 2-connected) graph.

Suppose, for a contradiction, that G has a separator S of size at most k. By adding vertices to S we may suppose that $|S|=k$. Let V_1 be a component of $G-S$ and $V_2 = V-(V_1\cup S)$. Let us define $G_i = G[V_i \cup S]$ and $G_i' = G_i + E(K(S))$, for $i \in \{1,2\}$, where $K(S)$ denotes the complete graph on S. Finally, let us define

$$a_i = d|V_i\cup S| - \binom{d+1}{2} - r_d(G_i), \quad \text{and} \quad b_i = d|V_i\cup S| - \binom{d+1}{2} - r_d(G_i'),$$

for $i \in \{1,2\}$. In other words, a_i and b_i are the d-dimensional degrees of freedom of G_i and G_i', respectively. By symmetry, we may assume that $a_1 - b_1 \le a_2 - b_2$.

We shall show that

$$r_d(G_i) \ge kd - \binom{d+1}{2} - (a_2-b_2) + 1, \quad i\in\{1,2\}, \tag{1}$$

and

$$r_d(G_1) + r_d(G_2) \le r_d(G) + kd - \binom{d+1}{2} - (a_2 - b_2). \tag{2}$$

By Lemma 6 this implies that $\mathcal{R}_d(G)$ has a vertical c-separation for some positive integer c with

$$c \le kd - \binom{d+1}{2} - (a_2-b_2) + 1 \le kd - \binom{d+1}{2} + 1,$$

thus contradicting our assumption on the vertical connectivity of $\mathcal{R}_d(G)$.

To show that Equation (1) holds, note that $r_d(G_i) = d|V_i| + kd - \binom{d+1}{2} - a_i$. Hence (using the fact that $a_1-b_1 \le a_2-b_2$) it suffices to show that

$$d|V_i| + kd - \binom{d+1}{2} - a_i \ge kd - \binom{d+1}{2} - (a_i - b_i) + 1,$$

or equivalently, that $d|V_i| - 1 \ge b_i$, for $i\in\{1,2\}$. Recall that b_i is the minimum number of new edges needed to make G_i' d-rigid. Since S is a clique in G_i', we may obtain a d-rigid supergraph of G_i' by ensuring that every vertex of $V_i - S$ has at least d neighbors in S. We can clearly do this using at most $d|V_i|$ edges; in fact, since S is a separator and G is connected, there is at least one edge connecting V_i and S, so adding $d|V_i|-1$ edges suffices.

It only remains to show that Equation (2) holds. We have

$$\begin{aligned} r_d(E_1) + r_d(E_2) &= d|V_1\cup S| - \binom{d+1}{2} - a_1 + d|V_2 \cup S| - \binom{d+1}{2} - a_2 \\ &= d|V| - \binom{d+1}{2} + kd - \binom{d+1}{2} - (a_1+a_2) \\ &= d|V| - \binom{d+1}{2} - (a_1+b_2) + kd - \binom{d+1}{2} - (a_2-b_2). \end{aligned}$$

Hence we only need to show that

$$d|V| - \binom{d+1}{2} - (a_1+b_2) \le r_d(G),$$

or in other words, that we can make G d-rigid by adding at most a_1+b_2 edges. Let A_1 and B_2 be edge sets such that $|A_1| = a_1, |B_2| = b_2$ and $G_1 + A_1$ and $G_2' + B_2$ are d-rigid. Now the $\mathcal{R}_d$-closure of $G+A_1+B_2$ contains $K(V_1\cup S)$, and in particular $K(S)$, as a subgraph. Thus it also contains the $\mathcal{R}_d$-closure of $G_2'+B_2$, which is $K(V_2\cup S)$. Hence it contains two complete graphs intersecting in $k\ge d$ vertices. It follows that the $\mathcal{R}_d$-closure of $G+A_1+B_2$ is d-rigid, and hence so is $G+A_1+B_2$, as required. □

Let G be a graph, and let k be a positive integer. We say that G is $[k,d]$-*rigid* if it is d-rigid and remains so after the deletion of any set of fewer than k vertices.

Proposition 9 *Let $G=(V,E)$ be a graph. If G is $[k,d]$-rigid, then $\mathcal{R}_d(G)$ is vertically k-connected.*

Proof: We prove by induction on k; the $k = 1$ case is trivial. Let us thus assume that $k \geq 2$, and let us suppose, for a contradiction, that $\mathcal{R}_d(G)$ has a vertical k'-separation (E_1, E_2) for some $k' < k$. Since E_1 and E_2 are both nonempty and G is connected, there exists a vertex $v \in V(E_1) \cap V(E_2)$. Note that $G - v$ is $[k-1, d]$-rigid, and hence by the induction hypothesis $\mathcal{R}_d(G - v)$ is vertically $(k-1)$-connected.

Let $E'_i = E_i - \partial_G(v)$ for $i \in \{1, 2\}$. Since G is $[k, d]$ rigid, we must have $d_G(v) \geq d + k - 1 \geq d + 1$, and by definition $d_{E_1}(v), d_{E_2}(v) \geq 1$. Hence we have

$$\begin{aligned} r_d(E'_1) + r_d(E'_2) &\leq r_d(E_1) + r_d(E_2) - (d+1) \\ &\leq r_d(G) + (k'-1) - d - 1 \\ &= r_d(G - v) + (k'-1) - 1. \end{aligned}$$

It follows that

$$r_d(E'_1) + r_d(E'_2) = r_d(G - v) + c - 1 \tag{3}$$

for some integer c with $1 \leq c \leq k' - 1$. If both $r_d(E'_1) \geq c$ and $r_d(E'_2) \geq c$, then (E'_1, E'_2) is a vertical c-separation of $\mathcal{R}_d(G - v)$, a contradiction. Thus $r_d(E'_1) \leq c - 1$ or $r_d(E'_2) \leq c - 1$; by symmetry, we may suppose that it is the former.

Now Equation (3) implies that $r_d(E'_1) = c - 1$ and $r_d(E'_2) = r_d(G - v)$. If $d_{E_2}(v) \geq d$, then $r_d(E_2) = r_d(E'_2) + d = r_d(G - v) + d = r_d(G)$, contradicting the fact that (E_1, E_2) is a vertical k'-separation. Hence $d_{E_2}(v) \leq d - 1$. It follows that

$$\begin{aligned} r_d(E_1) + r_d(E_2) &\geq d_{E_1}(v) + r_d(E'_2) + d_{E_2}(v) = r_d(G - v) + d_G(v) \\ &\geq r_d(G - v) + d + k - 1 = r_d(G) + k - 1 \\ &> r_d(G) + k' - 1. \end{aligned}$$

But this contradicts, again, the fact that (E_1, E_2) is a vertical k'-separation. □

By combining Theorem 3 and Proposition 9, we obtain the following corollary.

Corollary 10 *Let G be a graph, and let k be a nonnegative integer. If G is $(k + d(d+1))$-connected, then $\mathcal{R}_d(G)$ is vertically $(k+1)$-connected.*

We shall need the following folklore statement.

Lemma 11 *[3, Lemma 5.1] Let G and H be graphs without isolated vertices, and let $\psi : E(G) \to E(H)$ be a bijection that "sends stars to stars"; that is, for every $v \in V(G)$, there is a vertex $v' \in V(H)$ such that $\psi(\partial_G(v)) = \partial_H(v')$. Then ψ is induced by a graph isomorphism.*

We are now ready to prove our main results.

Proof of Theorem 4: Suppose that $G = (V, E)$ is $d(d+1)^2$-connected, let $H = (V', E')$ be a graph without isolated vertices, and let $\psi : E \to E'$ be an isomorphism of $\mathcal{R}_d(G)$ and $\mathcal{R}_d(H)$. Our goal is to show that ψ is induced by a graph isomorphism.

Fix $v \in V$ and consider $F = E - \partial_G(v)$. Let H_0 denote the subgraph of H induced by $\psi(F)$; note that $\mathcal{R}_d(G - v)$ and $\mathcal{R}_d(H_0)$ are isomorphic. Since both G and $G - v$ are $(d(d+1)^2 - 1)$-connected, G and $G - v$ are both d-rigid by Theorem 3, and $\mathcal{R}_d(G)$ and $\mathcal{R}_d(G - v)$ are both vertically $d^2(d+1)$-connected by Corollary 10. It follows from Proposition 8 and a short computation that H and H_0 are both $d(d+1)$-connected, and hence by Theorem 3 they are both d-rigid.

This means that

$$d|V(H_0)| - \binom{d+1}{2} = r_d(\psi(F)) = r_d(F) = r_d(E) - d = r_d(E') - d = d(|V'| - 1) - \binom{d+1}{2},$$

and thus $|V(H_0)| = |V'| - 1$. Also, since F is the edge set of an induced subgraph of G, it is closed in $\mathcal{R}_d(G)$, and hence $\psi(F)$ is closed in $\mathcal{R}_d(H)$. It follows that H_0 is an induced subgraph of H, since otherwise (by the fact that H_0 is d-rigid) we could add edges induced by $V(H_0)$ in H to $\psi(F)$ without increasing its rank.

To summarize, $\psi(F)$ is the edge set of an induced subgraph of H on $|V'| - 1$ vertices, which implies that it is the complement of a vertex star. This shows that ψ maps complements of vertex stars to complements of vertex stars. Since ψ is a bijection, it also follows that it maps vertex stars to vertex stars. Now Lemma 11 implies that ψ is induced by a graph isomorphism, as desired. □

Proof of Theorem 5: Our goal is to show that G is $d(d+1)^2$-connected; then the statement follows from Theorem 4. By Proposition 8, it suffices that $\mathcal{R}_d(G)$ is vertically k-connected, where

$$k = d(d(d+1)^2 - 1) - \binom{d+1}{2} + 2 = (d(d+1))^2 - d - \binom{d+1}{2} + 2 \leq (d(d+1))^2.$$

Thus G is indeed $d(d+1)^2$-connected, and hence $\mathcal{R}_d$-reconstructible, as required. $\square$

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